International Journal of Theoretical Physics, Vol. 45, No. 1, January 2006 (© 2006)
DOI: 10.1007/s10773-005-9007-x

Solution of the Dirac Equation in Expanding Universes

Antonio Zecca[1,2]

Received April 7, 2005; accepted November 24, 2005
Published Online: April 27, 2006

Exact solutions of the Dirac equation in the Robertson–Walker space-time are obtained by an elementary separation method that represents a straightforward improvement of previous results. The radial equations are integrated by reporting them to hypergeometric equations. The separated time equations are solved exactly for three models of universe expansion and integrated by series in a case of the standard cosmological model. The integration of both radial and time equations represents an improvement of previous results.

KEY WORDS: Dirac equation; curved expanding universes; exact solutions.

1. INTRODUCTION

The Dirac equation and its quantization in curved space-time is of great interest owing to the behavior of spin 1/2 particles in astrophysics and cosmology. The relevance of this appears in the general discussion on the interaction of neutrinos ($m = 0$) and spherically symmetric gravitational fields performed by Brill and Wheeler (1957). The explicit solution of the Dirac equation in curved space-time is crucial in expanding universes and in particular in the Robertson–Walker space-time that is the base of the standard cosmological model. From the mathematical point of view the Dirac equation can be formulated in at least two equivalent but different ways: the four dimensional spin connection formulation (Weinberg, 1972) and the two two spinor one (e.g., Penrose and Rindler, 1984). To the first class there belongs the paper by Parker (1972) whose interest is the production of spin 1/2 particles as a result of an expanding universe. In the paper by Barut and Duru (1987) (where earlier references on the subject are reported) the spin connection point of view is assumed and the Dirac equation is exactly solved for spatially flat Robertson–Walker space-time in three

[1] Gruppo Nazionale per la Fisica Matematica, Italy.
[2] Dipartimento di Fisica dell' Universita' and INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy; e-mail: antonio.zecca@mi.infn.it.

meaningfull models of expanding universes. On the other side the spinor formulation was the basic context in which Chandrasekhar (1983) succeeded to separate the Dirac equation in the Kerr metric that also includes the Schwarzschild metric as a special case. Chandrasekhar's method was extended to the Robertson–Walker space-time by the author of this paper who did not know the work by Barut and Duru (1987) as well as many of the references therein. The radial Dirac equation was exactly solved in the open, closed and flat space-time case (Zecca, 1996), the separated time equation integrated in the cases of the standard cosmological model (Zecca, 1998) and the normal modes determined (Montaldi and Zecca, 1998). The Dirac equation has been separated also in Tolman-Bondi models (Zecca, 2000).

In the present paper another solution of the Dirac equation is proposed. The starting point is the spin connection formulation of the Dirac equation that leads, as already checked in the general as well as in the Robertson–Walker metric (e.g., Zecca, 2002, 2003) to exactly the same differential equations obtained from the spinor formulation. The equations are then separated by an elementary separation method that represents an improvement of the method used in Zecca, 1996. The radial equations are integrated exactly in every case and the results are in line with those of Barut and Duru (1987) and Huang (2005). The separated time evolution, that has been integrated by series in Zecca, 1998 for the standard cosmological evolution, is here characterized by two a priori different time equations. These equations are made to essentially coincide by a simple condition on the separation constants. They are exactly integrated in the linear, exponential and radiation dominated universe expansion, three models of universe evolution already considered by Barut and Duru (1987). Finally, also a case of matter dominated universe is discussed, the corresponding time equation reduced to a simple form and easily integrated by series.

2. THE DIRAC EQUATION AND ITS SEPARATION

The study of the Dirac equation in Robertson–Walker space-time of metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - R(t)^2 \left[\frac{dr^2}{1 - ar^2} + r^2 (d\theta^2 + \sin\theta^2 d\varphi^2) \right], \qquad a = 0, \pm 1. \tag{1}$$

is done by starting from the spin connection form of the equation that reads

$$\gamma^\mu (\partial_\mu - \Omega_\mu)\psi = i m_0 \psi \tag{2}$$

where ψ is a four dimemsional spinor, γ^μ, Ω^μ the "curvature dependent" gamma matrices and the spin connection vector, respectively. By choosing the tetrad e_b^μ (b labels the vectors of the tetrad, μ the components of the vectors) to be

$$e_b^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \sqrt{1-ar^2}/R & 0 & 0 \\ 1 & -\sqrt{1-ar^2}/R & 0 & 0 \\ 0 & 0 & 1/rR & i/(rR\sin\theta) \\ 0 & 0 & 1/rR & -i/(rR\sin\theta) \end{pmatrix} \tag{3}$$

the scheme can be studied as in Zecca (2003) by representing the spinor ψ by two components spinors

$$\psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix}, \quad \eta = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \chi = \begin{pmatrix} -G_2 \\ G_1 \end{pmatrix}. \tag{4}$$

Moreover, the spin connection vector can be calculated, the gamma matrices represented in the two components spinor formalism based on the tetrad (3) so that the Dirac Eq. (2) can finally, be explicited so to obtain (Zecca, 2002, 2003)

$$\begin{aligned} (\alpha - \delta^\star)G_2 + (\Delta + \epsilon + \mu)G_1 &= im_0 F_1 \\ (D + \epsilon - \rho)G_2 - (\delta - \alpha)G_1 &= im_0 F_2 \\ (\delta - \alpha)F_1 + (\Delta + \epsilon + \mu)F_2 &= im_0 G_2 \\ (D + \epsilon - \rho)F_1 + (\delta^\star - \alpha)F_2 &= im_0 G_1 \end{aligned} \tag{5}$$

The surviving spin coefficients and directional derivatives appearing in Eq. (5) are given by

$$\begin{aligned} &\rho = -(r\dot{R} + \sqrt{1-ar^2})/(rR\sqrt{2}); \quad \mu = (r\dot{R} - \sqrt{1-ar^2})(rR\sqrt{2}) \\ &\alpha = -\cot\theta/(2rR\sqrt{2}); \qquad \epsilon = \dot{R}/(2R\sqrt{2}) \\ &D = e_1^\mu \partial_\mu, \quad \Delta = e_2^\mu \partial_\mu \quad \delta = e_3^\mu \partial_\mu, \quad \delta^\star = e_4^\mu \partial_\mu. \end{aligned} \tag{6}$$

Apart from a factor $\sqrt{2}$ in the mass of the particle (that depends on the definitions of the gamma matrices), the equation (10) coincides with the explicitation of the Dirac equation in spinor form (Zecca, 2003). The Eq. (6) can be separated in a standard way by setting

$$\begin{aligned} (F_1,\ F_2) &= \frac{\exp(im_0\varphi)}{rR}(H_1(r,t)S_1(\theta),\ H_2(r,t)S_2(\theta)) \\ (G_1,\ G_2) &= \frac{\exp(im_0\varphi)}{rR}(H_2(r,t)S_1(\theta),\ H_1(r,t)S_2(\theta)) \end{aligned} \tag{7}$$

The separated angular equation imply for the separation constant λ the value $\lambda^2 = (l + 1/2)^2,\ l = |m|, |m| + 1, |m| + 2, \ldots,$ and $S_{1lm}(\theta),\ S_{2lm}(\theta)$

essentially given by the Jacobi polynomials for $|m| \geq 1$ while for $m = 0$, $l = 0, 1, 2, \ldots$, $\lambda^2 = (l+1)^2$ and the $S_{ilm}(\theta)$'s $(i = 1, 2)$ are essentially the Tchebychef polynomials of second kind (Montaldi and Zecca, 1994).

For what concerns the separated equations in the r, t variables, two of them result to duplicate the other two so that one is left with

$$DH_1 + \epsilon H_1 = \left(i\, m_0 - \frac{\lambda}{rR\sqrt{2}} H_2\right)$$
$$\Delta H_2 + \epsilon H_2 = \left(i\, m_0 + \frac{\lambda}{rR\sqrt{2}} H_2\right). \tag{8}$$

By setting

$$H_1 = f(r,t) + g(r,t) \quad H_2 = f(r,t) - g(r,t) \tag{9}$$

and using the explicit form of the directional derivatives and spin coefficients in (6), the equations for f, g result to be

$$\dot{f} + \frac{\sqrt{1-ar^2}}{R} g' + f\left(\frac{\dot{R}}{2R} - i\, m_0\sqrt{2}\right) - \frac{\lambda}{rR} g = 0$$
$$\dot{g} + \frac{\sqrt{1-ar^2}}{R} f' + g\left(\frac{\dot{R}}{2R} + i\, m_0\sqrt{2}\right) + \frac{\lambda}{rR} f = 0 \tag{10}$$

(prime and dot means ∂_r, ∂_t, respectively). By further setting

$$f = F(r)T(t), \qquad g = G(r)S(t) \tag{11}$$

the r, t dependence can be finally, separated into the equations

$$\sqrt{1-ar^2}\, G' - \frac{\lambda}{r} G = -k_1 F \tag{12}$$

$$\sqrt{1-ar^2}\, F' + \frac{\lambda}{r} F = -k_2 G, \quad (a = 0, \pm 1) \tag{13}$$

$$\dot{T}\, R + T\left(\frac{\dot{R}}{2} - i\, m_0 R\sqrt{2}\right) = k_1 S \tag{14}$$

$$\dot{S}\, R + S\left(\frac{\dot{R}}{2} + i\, m_0 R\sqrt{2}\right) = k_2 T \tag{15}$$

k_1, k_2 the separation constants of Eqs. (10). Therefore, the equations for F, G and T, S are then, respectively

$$(1-ar^2)F'' - arF' - \left(\frac{\lambda\sqrt{1-ar^2} + \lambda^2}{r^2} + k_1 k_2\right) F = 0 \tag{16}$$

$$(1-ar^2)G'' - arG' + \left(\frac{\lambda\sqrt{1-ar^2}-\lambda^2}{r^2} - k_1k_2\right)G = 0 \tag{17}$$

$$\ddot{T} + 2\frac{\dot{R}}{R}\dot{T} + \left(\frac{\ddot{R}}{2R} + \frac{1}{4}\frac{\dot{R}^2}{R^2} - im_0\sqrt{2}\frac{\dot{R}}{R} + 2m_0^2 - \frac{k_1k_2}{R^2}\right)T = 0 \tag{18}$$

$$\ddot{S} + 2\frac{\dot{R}}{R}\dot{S} + \left(\frac{\ddot{R}}{2R} + \frac{1}{4}\frac{\dot{R}^2}{R^2} + im_0\sqrt{2}\frac{\dot{R}}{R} + 2m_0^2 - \frac{k_1k_2}{R^2}\right)S = 0 \tag{19}$$

Due to the arbitrariness of $k_1,\ k_2$ if one chooses

$$k_1 = k_2 = k = k^\star \tag{20}$$

not only Eq. (17) follows from Eq. (16) by the substitution $\lambda \to -\lambda$ but also the solution S can be obtained from the solution T through $S = T^\star$.

3. SOLUTION OF THE RADIAL EQUATION

As mentioned, it suffices to study one only radial equation. For the solution it is usefull to distinguish according to the curvature of the space-time.

Flat space-time case: $a = 0$. In the present case the Eq. (17) reads

$$G'' + G\left(\frac{\lambda-\lambda^2}{r^2} - k^2\right) = 0 \tag{21}$$

By passing to the function $z(\xi)$ defined by $G = r^\lambda e^{kr} z$ in the independent variable $\xi = -2kr$, the Eq. (21) reduces to a confluent hypergeometric equation. One finds then

$$G_1 = r^\lambda\, e^{kr}\phi(\lambda; 2\lambda; -2kr) \tag{22}$$

(A second solution can be obtained in a standard way (Abramovitz and Stegun, 1970) but it has not a simple form in the case in which λ is an integer number).

Curved space-time case: $a = \pm 1$. By using the variable $\rho = \sqrt{1-ar^2}$, the Eq. (16) becomes

$$F'' + \frac{\rho}{\rho^2-1}F' + \frac{(\rho-1)^2k^2/a - \lambda\rho - \lambda^2}{(1-\rho^2)^2}F = 0 \quad (a = \pm 1) \tag{23}$$

that can be reported to a hypergeometric equation in the variable $\xi = (1-\rho)/2$ for the function z defined by $F = [(1-\xi)/\xi]^{\lambda/2} z(\xi)$. One finds then that two possible solutions are given by

$$F_1 = \left(\frac{1+\sqrt{1-ar^2}}{1-\sqrt{1-ar^2}}\right)^{\frac{\lambda}{2}} F\left(\frac{ik}{\sqrt{a}}; -\frac{ik}{\sqrt{a}}; \frac{1}{2}-\lambda; \frac{1-\sqrt{1-ar^2}}{2}\right)$$

$$F_2 = \left(\frac{r}{2}\right)^{\lambda} \left(\frac{1-\sqrt{1-ar^2}}{2}\right)^{\frac{1}{2}} F\left(\frac{ik}{\sqrt{a}} + \lambda + \frac{1}{2}; \frac{-ik}{\sqrt{a}}\right.$$
$$\left. + \lambda + \frac{1}{2}; \lambda + \frac{3}{2}; \frac{1-\sqrt{1-ar^2}}{2}\right) \tag{24}$$

where $a = \pm 1$. (Compare with Zecca (1996) and Huang (2005)).

4. TIME EQUATION: MODELS OF EXPANDING UNIVERSES

A separated Dirac equation was already considered by the author (Zecca, 1998). There the time equation was integrated by series in the cases of the time evolution of the standard cosmological model. Here the time equation will be first integrated in the three expansion models considered by Barut and Duru (1987) and then in a case of the standard cosmology.

Linear universe expansion model. As mentioned by Barut and Duru (1987), the expansion law $R = Ht$ was already discussed by Schrödinger (1939). It represents the curvature dominated expansion governed by the Freedman equation for a fluid whose state equation is given by $p = -\rho/3$. (e.g., Kolb and Turner, 1990). The Eq. (18) becomes now

$$\ddot{T} + \frac{2}{t}\dot{T} + \frac{1/4 - k^2/H^2 - im_0\sqrt{2}\,t + 2m_0^2 t^2}{t^2} T = 0. \tag{25}$$

By setting $T = t^{\alpha} e^{im_0\sqrt{2}t}\eta(t)$ and successively $\xi = -2im_o\sqrt{2}\,t$, the equation for η results to be

$$\xi\eta'' + (2\alpha + 2 - \xi)\eta' - \left(\alpha + \frac{1}{2}\right)\eta = 0 \tag{26}$$

with α defined by $2\alpha + 1 = \sqrt{1 - 4(k/H)^2}$. Two independent solutions are then

$$T_1 = t^{\alpha} e^{im_0\sqrt{2}\,t}\ \phi\left(\alpha + \frac{1}{2}; 2\alpha + 2;\ -2im_0\sqrt{2}\ t\right)$$
$$T_2 = t^{-1-\alpha} e^{im_0\sqrt{2}\,t}\ \phi\left(-\alpha - \frac{1}{2};\ -2\alpha;\ -2im_0\sqrt{2}\,t\right) \tag{27}$$

$\phi(a; b; x)$ the confluent hypergeometric function.

Exponential expansion. The law $R = e^{Ht}$ is a characteristic expansion of an inflationary universe if $H > 0$ and even of a "supercooling" fase of an inflationary universe if $H < 0$ (Kolb and Turner, 1990). To integrate the time equation it is usefull to introduce the variable $z = (k/H)\,e^{-Ht}$ for which the expansion law becomes $R = k/(Hz)$ and the Eq. (18) reads

$$T'' - \frac{1}{t}T' + \frac{3/4 - im_0\sqrt{2}/H + 2(m_0/H)^2 - z^2}{z^2} T = 0 \tag{28}$$

(prime means here d/dz). By setting $T = z^{\alpha} f,\ \ \alpha = 1/2 - im_0\sqrt{2}/H$, into Eq. (28) and then $\xi = -2z$, the equation for f is

$$\xi f'' + (2\alpha - 1 - \xi) f' - \left(\alpha - \frac{1}{2}\right) f = 0. \tag{29}$$

Therefore, the expression

$$T = \left(\frac{k}{H}e^{-Ht}\right)^{\alpha} e^{(k/H)e^{-Ht}}\ \phi\left(\alpha - \frac{1}{2}; 2\alpha - 1;\ -2(k/H)\, e^{-Ht}\right) \tag{30}$$

is a solution of Eq. (18). A second solution can be obtained from a standard second solution of the confluent hypergeometric equation.

Radiation dominated standard cosmology. In this model the expansion law is of the form $R = a_0\sqrt{t},\ a_0$ a constant number (e.g. Kolb and Turner, 1990). The Eq. (18) reads now

$$\ddot{T} + \frac{1}{t}\dot{T} + \frac{2m_0^2 t^2 - \left(im_0/\sqrt{2} + k^2/a_0^2\right)t - 1/16}{t^2} T\ = 0. \tag{31}$$

One can reduce to a confluent hypergeometric equation by setting $T = t^{-1/4} e^{im_0\sqrt{2}\,t} g(t)$ and then $\xi = -2\chi\, t$:

$$\xi\, g'' + \left(\frac{1}{2} - \xi\right) g' - \frac{ik^2}{2\sqrt{2}a_0^2 m_0^2}\, g = 0. \tag{32}$$

A solution is now

$$T\ = t^{-1/4}\, e^{im_0\sqrt{2}\,t}\, \phi\left(\frac{ik^2}{2\sqrt{2}a_0^2 m_0^2}; \frac{1}{2};\ -2im_0\sqrt{2}\,t\right) \tag{33}$$

Also here a second solution follows from standard properties of the confluent hypergeometric equation (Abramovitz and Stegun, 1970).

A standard cosmological expansion. Suppose now the space-time filled with comparable contribution of matter and radiation to the energy density but with negligible curvature. For time much greater then the equilibrium time, the standard cosmology predicts an expansion law of the form $R = a_0\, t^{2/3},\ \ a_0$ constant. Correspondingly the Eq. (19) can be integrated by series introducing the independent variable

$$\tau = \int_0^t \frac{dt'}{R(t')}. \tag{34}$$

The expansion law in the variable τ is then $R = (a_0^3/9)\tau^2 = \alpha\tau^2$. The Eq. (19) reads (prime mens d/dz)

$$S'' + \frac{2}{\tau}S' + \left[2m_0^2\alpha^2\,\tau^4 + 2im_0\sqrt{2}\alpha\,\tau - k^2\right]S = 0 \tag{35}$$

and by setting $y = \tau\,S$,

$$y'' + y[a\tau^4 + ib\tau - k^2] = 0 \tag{36}$$

$a = 2m_0^2\alpha^2, \;\; b = 2\sqrt{2}\alpha m_0$. By inserting the expression $\sum_0^\infty c_n\tau^n$ into Eq. (36), the first terms of two independent integrals corresponding to the choices $(c_0, c_1) = (0, 1)$ and $(c_0, c_1) = (1, 0)$, respectively, can be determined:

$$\begin{aligned} y_1 &= \tau + \frac{k^2}{6}\tau^3 - \frac{ib}{12}\tau^4 + \frac{k^4}{120}\tau^5 - \frac{ibk^2}{120}\tau^6 + \cdots \\ y_2 &= 1 + \frac{k^2}{2}\tau^2 - \frac{ib}{6}\tau^3 + \frac{k^2}{24}\tau^4 - \frac{ibk^2}{30}\tau^5 + \frac{k^2+2b^2}{720}\tau^6 + \cdots \end{aligned} \tag{37}$$

The solution provides a simplification of a similar result given in Zecca, 1998.

REFERENCES

Abramovitz, W. and Stegun, I. E. (1970). *Handbook of Mathematical Functions*, Dover, New York.

Barut, A. O. and Duru, I. H. (1987). Exact solutions of the Dirac equation in spatially flat Robertson–Walker space-time. *Physical Review D* **36**, 3705.

Brill, D. R. and Wheeler, J. A. (1957). Interaction of neutrinos and gravitational fields. *Review of Modern Physics* **29**, 465.

Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Oxford University Press, New York.

Huang, X.-B. (2005). Exact solutions of the Dirac equation in Robertson–Walker space-time. arXiv: gr-qc/0501077.

Kolb, E. W. and Turner, M. S. (1990). *The Early Universe*, Addison Wesley, New York.

Montaldi, E. and Zecca, A. (1994). Neutrino Wave equation in the Robertson–Walker Geometry. *International Journal of Theoretical Physics* **3**, 1053.

Montaldi, E. and Zecca, A. (1998). Second quantization of the Dirac field: Normal modes in the Robertson–Walker space-time. *International Journal of Theoretical Physics* **37**, 995.

Parker, L. (1972). Backscattering caused by the expansion of the universe. *Physical Review D* **5**, 2905.

Penrose, R. and Rindler, W. (1984). *Spinors and Space-Time*, Cambridge University Press, Cambridge.

Scrödinger, E. (1939). The proper vibrations of the expanding universe. *Physica, Sec. A.* **6**, 899.

Weinberg, S. (1972). *Gravitation and Cosmology*, John Wiley, New York.

Zecca, A. (1996). The Dirac equation in the Robertson–Walker space-time. *Journal of Mathematical Physics* **37**, 874.

Zecca, A. (1998). Dirac equation in standard cosmology: Time evolution. *Il Nuovo Cimento B* **113**, 915.

Zecca, A. (2000). Dirac equation in Lemaitre-Tolman-Bondi models. *General Relativity and Gravitation* **32**, 1197.

Zecca, A. (2002). Dirac equation in space-time with torsion. *International Journal of Theoretical Physics* **41**, 421.

Zecca, A. (2003). The spin connection, the Dirac and the Einstein Dirac equation in Robertson–Walker space-time. *Il Nuovo Cimento B* **118**, 693.